PH1150 Teamwork Project: Deterministic Chaos - Strange Attractors

Farbod Rassouli Sayyed, Tomoyuki Jinno, Aleksas Girenas, Jason Duong

October 3, 2020

1 Abstract

We researched and analysed the behaviour of the Lorenz attractor by using the Lorenz equations on Mathematica[1] and Grapher[2] on MacOS. By first understanding the idea of chaos and an 'attractor' we plotted graphs ranging from simple attractors to the more complex Lorenz Attractor. We concluded that the Lorenz system of equations will always have an attractor. Upon placing particles in the system we saw that it is a global attractor when $\rho < 1$ and the particle will always tend to a point; if $\rho > 1$ it is a strange attractor and the particle will always tend to a surface. In particular we investigated the time taken for two trajectories to diverge significantly for a Lorenz attractor as a function of the closeness of the start points: to find the dependence of the initial starting positions on the time taken to diverge significantly we wrote a script in Mathematica that computes the time taken for the two norm of the trajectories to reach a unit length of 1 for a range of small displacements in the y-axis. We showed that the initial displacement of the point s and time t taken to diverge has a negative exponential relationship. To prove the chaotic behaviour we first researched the relationship to fractals and by finding its fractal dimension using the Hausdorff equation to be 2.06 ± 0.01 we found that the attractor is strange so it may have chaotic behaviour; we found that the Lyapunov exponent could be used to prove if the Lorenz attractor was chaotic by determining how sensitive the Lorenz attractor is to initial conditions. It is very sensitive to the initial conditions as the exponent was $\lambda = 0.913616$ and so it is chaotic as it is a positive value. Non-chaotic behaviour was shown when $\rho < 1$ and we found the Lyapunov exponent to be $\lambda = -1.27387$ supporting the argument that it is not chaotic for this value as the exponent is negative.

2 Introduction

Deterministic chaos, or more simply chaos, and strange attractors like the Lorenz attractor are part of a subject known as Dynamics.[3] This is the subject that deals with change and systems that evolve in time. Chaos theory represents a set of techniques for analysing dynamic systems that follow apparently simple rules leading to behaviours which depend only upon their initial conditions and yet are very sensitive to slight changes in the input. Examples of dynamic systems include: fluid dynamics; population analysis; economic behaviour; the weather. Despite knowing a great deal about all the elements that constitute physical patterns we have only been able to predict the outcomes of these systems with modest accuracy. The theory of chaos was summarized by the American meteorologist Edward Lorenz as:

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.[4]

Chaos theory may provide insight into the workings of these complex dynamic systems.[5] When we look within the apparent randomness of chaotic complex systems, there are underlying patterns, loops, repetition, self-similarity, fractals and self-organization.[6] By understanding how these underlying patterns work we can have a greater understanding of chaotic systems. Advances in understanding these systems could lead to having a greater understanding of the physical world and designing technology that interfaces with complex dynamic systems.

In 1961, Lorenz found another example of chaotic behaviour. While researching convection Lorenz developed a system for predicting the weather based on 12 differential equations. The equations represented the factors we know to affect weather patterns, including pressure, temperature, and wind velocity. When he computed these equations Lorenz found that by making very small changes in the initial numbers used in these equations, he could produce wildly different results.[7]

Then in 1963, Lorenz developed a simplified mathematical model for atmospheric convection [8] which is a system of three ordinary differential equations, known as the Lorenz equations, showing velocities in three dimensional space:

$$\frac{dx}{dt} = \sigma(y - x), \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

By changing the values of the constants σ, ρ and β in equations 1, 2 and 3 we can model a range of systems that are both complex as in Fig.1 and complex as in Fig.2 which is the Lorenz attractor.

Figure 1: Simple attractor plotted using Grapher software.

Figure 2: Lorenz attractor plotted using Grapher software.

The Lorenz attractor is one of many examples of a strange attractor which exhibit chaotic behaviour.[14]

We used Mathematica for mathematical analysis and graphing of properties and used the Grapher software (MacOS) for visual graphics of the systems as it had a more clear and dynamic visual representation of attractors.

3 Analysis of Attractors

3.1 Simple Attractors

Before we can understand the behaviour of the Lorenz Attractor and its workings we must define what an attractor is: an attractor is a state or behavior toward which a dynamic system tends to evolve, represented as a point or orbit in the system's phase space. We consider the system to be stable when it is within the attractor's area[9]

A simple attractor was developed by using the Lorenz system of equations 1, 2 and 3 and for values of $\rho < 1$ there is only one stable point, which is at the origin. This point corresponds to no convection. All orbits converge to the origin, which is a global attractor, when $\rho < 1$.

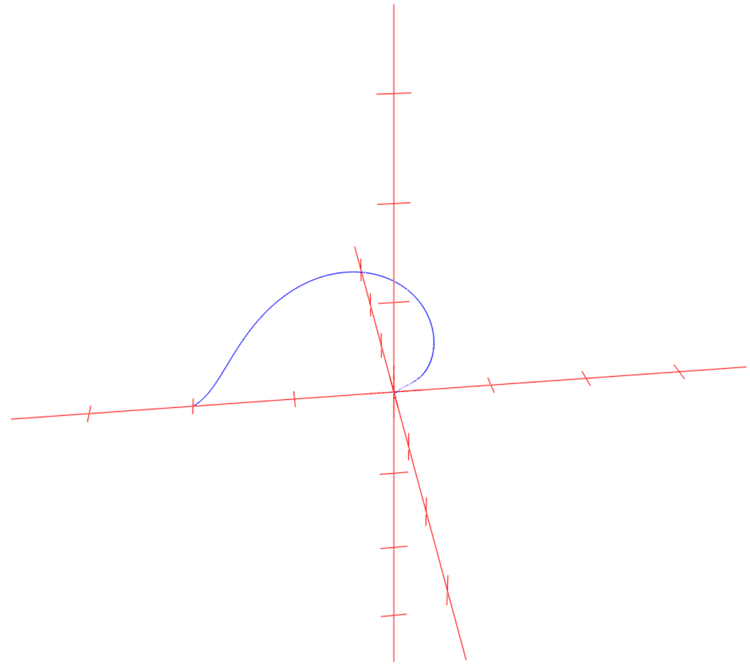

Figure 3: Simple attractor stable at the origin when $\rho < 1$ using Grapher software.

As we increased the value of ρ to $\rho = 1$ a bifurcation occured. A bifurcation is when a small smooth change made to the parameter values of a system cause a sudden topological change in its behaviour.[17].

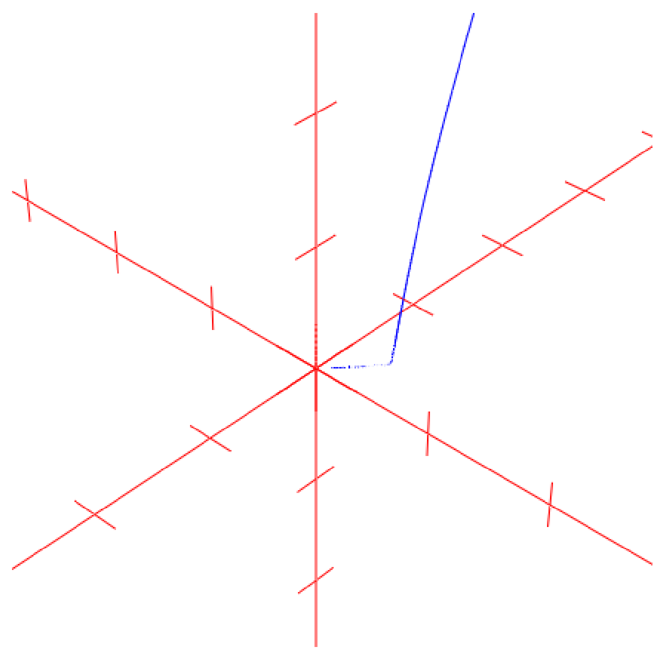

Figure 4: Close up of a bifurcation when $\rho = 1$ for a simple attractor near the origin using Grapher software. (The general shape of the attractor is similar to Fig.1).

When $\rho > 1$ the system converges to a point on a surface in space and the system is considered to be a fixed point and not chaotic or unpredictable.

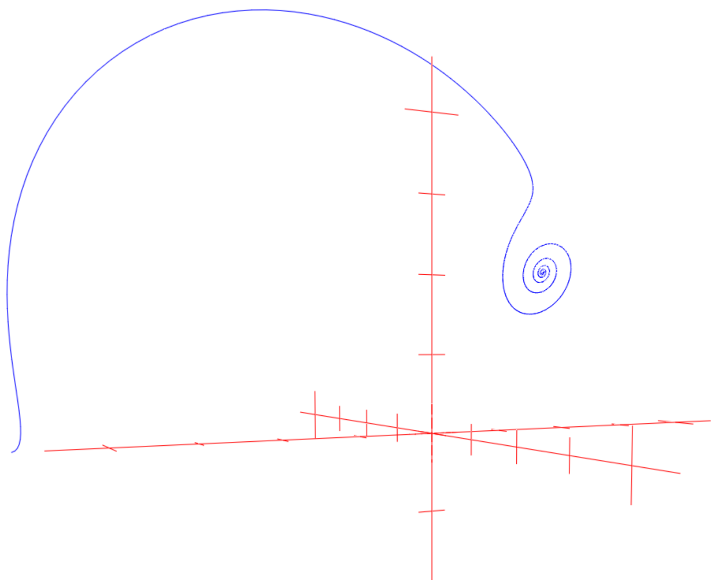

Figure 5: An attractor converging to a point in space when $\rho > 1$ using Grapher software.

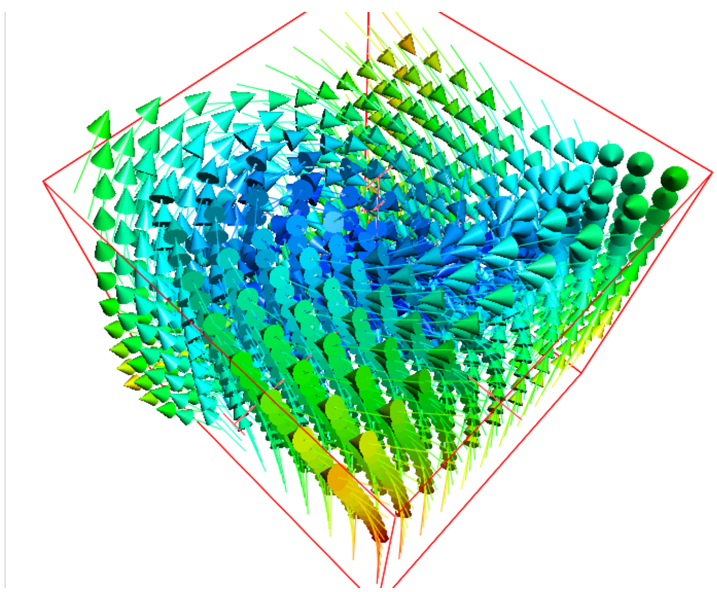

Figure 6: A vector field of an attractor converging to a point in space when $\rho > 1$ analogous to Fig.5 using Grapher.

As demonstrated attractors act much like orbits towards a certain point.

3.2 Strange Attractors - The Lorenz attractor

The Lorenz attractor is formed from the system of equations 1, 2 and 3. When $\rho > 1$ the system will become a strange attractor with 2 critical points in 3 dimensions. The points define the surfaces on which the attractor revolves and are described by equations 4 and 5.[15]

$$(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)) \tag{4}$$

$$(\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1) \tag{5}$$

When the particle is placed in the system it will rotate around the points in equations 4 and 5 when it is in its stable state. This is dependent on β and ρ and so changing the constants that define the surfaces of the attractor will change the way in which a particle will rotate; this changes the shape of the attractor. We demonstrated that the equation describes the critical points by plotting the trajectory of the system and the critical points on the same plot.

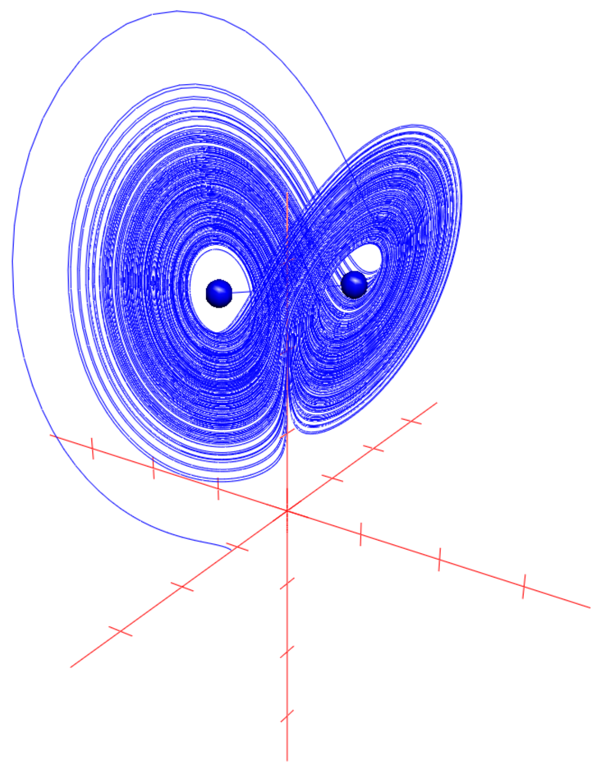

Figure 7: Graph plotted by Grapher to show the critical points for a Lorenz attractor.

The mid point of the critical points are always the same for any constant since the two equations are symmetric in x, y and z. Distance between critical points have been investigated by plotting the distance between the critical points over change in ρ and β shown in Fig.8, Fig.9. Fig.8 shows that the distance between the two critical points become zero when rho is 1 and no real solution exist when it is less than 1.

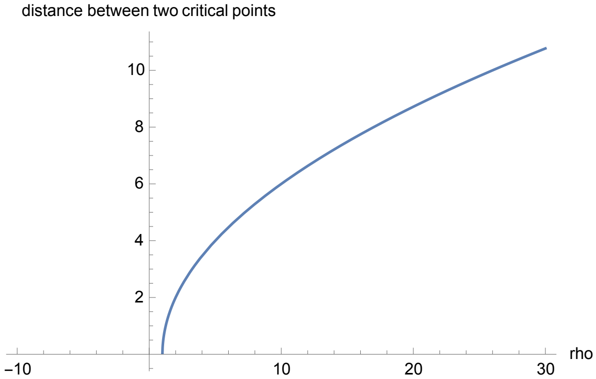

Figure 8: Graph plotted on Mathematica showing the distance dependence on ρ whilst β is kept constant at 1

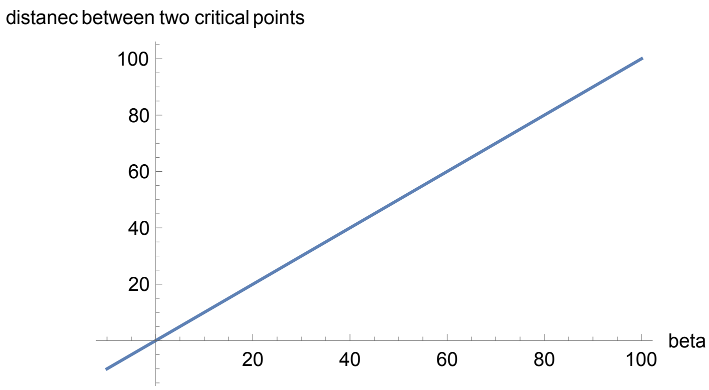

Figure 9: Graph plotted on Mathematica showing the distance dependence on β whilst ρ is kept constant at 2

Stable convection can only occur for positive ρ if $\sigma > \beta + 1$. At the critical value of $\sigma = \beta + 1$, both equilibrium points lose stability through a Hopf bifurcation.

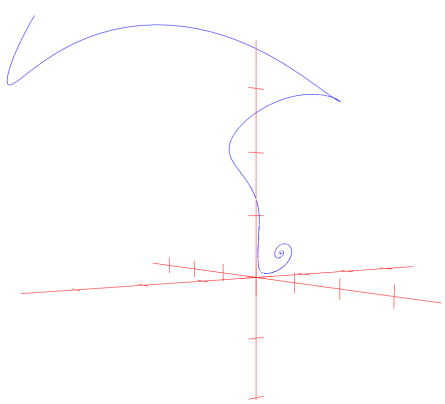

Figure 10: Graph plotted on Grapher demonstrating the sudden change caused by Hopf bifurcation

When we increased the distance that the particle is placed away from the attractor we found that it spiraled until it reached the area of influence of the attractor. The particle is considered to be unstable until it reaches the area of the attractor. This is demonstrated by Fig.10.

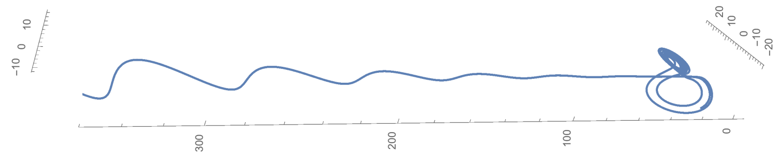

Figure 11: Graph plotted on Mathematica showing the spiraled effect of the trajectory before reaching the attractor surface

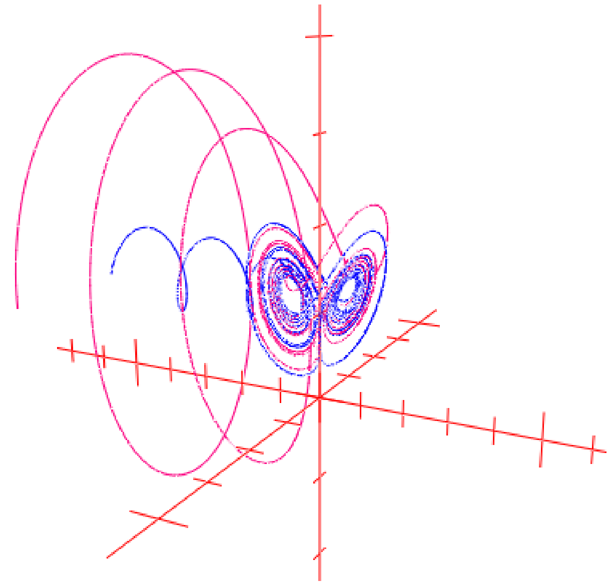

Figure 12: Grapher plot to show the unstable state of the Lorenz attractor where particles start at different points.

We concluded that the Lorenz system of equations 1, 2 and 3 will always have an attractor. Upon placing particles in the system we saw that if the system is a global attractor then the particle will always tend to a single point; if it is a strange attractor then the particle will always tend to a surface.

4 Chaos

4.1 Fractal Relationship to the Lorenz Attractor

By definition chaotic systems are systems that are sensitive to their initial state. To understand the chaotic nature we first looked at fractals. A fractal is an object that has the following properties: a never ending pattern; infinitely complex; self similarity; having a fractal dimension. This means zooming in on a small region of a fractal leaves you looking at the same shape you started with and smaller parts of the fractal can look exactly the same as the whole fractal shape [10]. Famous fractal patterns include the Sierpinski triangle and Mandelbrot set. Understanding the fractal dimension was vital to understanding the chaos of the Lorenz attractor: a fractal dimension is a dimension that is in-between dimensions so not 1D, 2D or 3D but in-between i.e.1.3D. There are many fractals in nature such as the brain, trees and lungs. A fractal dimension can be computed using Hausdorff's dimension equation 7 [11].

$$D = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} \tag{6}$$

D = Kaplan-Yorke dimension

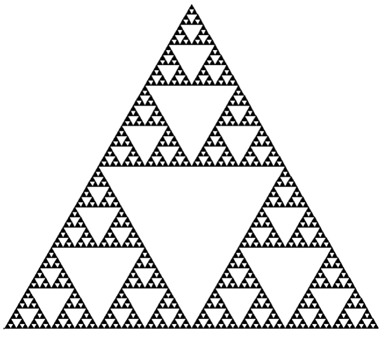

Figure 13: Example of a fractal - Sierpinski triangle

λ_i = Lyapunov exponent for dimension i

$$D = \frac{Log(N)}{Log(\frac{1}{r})} \tag{7}$$

D = the dimension
N = the number of smaller copies
r = the scale factor that the smaller copies are scaled down by.

The Lorenz attractor has a Kaplan-Yorke dimension of 2.06 $\pm$0.01 which means that the attractor is a fractal and therefore it is strange. Therefore the Lorenz Attractor exhibits properties of fractals.

4.2 Chaotic property

In the Lorenz system of equations the trajectory of the particle in the system has a sensitive dependence on the initial starting positions. Therefore we investigated the time taken for two trajectories to diverge significantly for a Lorenz attractor as a function of the closeness of the start points. We first demonstrated that the system is sensitive to it's initial position by simulating the divergence of two trajectories, with initial positions displaced by 0.0001, using Mathematica as shown in Figure[14].

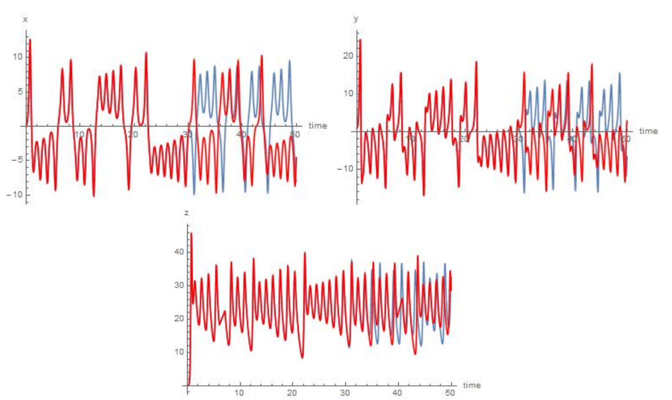

Figure 14: Two trajectories diverging in the x,y and z axis after t=30. The initial position of the blue particle:[0,1,0] and the red particle:[0,1.0001,0]. System constants are $\rho = 26.5$, $\sigma = 3$ and $\beta = 1$, using Mathematica.

To find the dependence of the initial starting positions on the time taken to diverge significantly we wrote a script in Mathematica that computes the time taken for the two norm of the trajectories to reach a unit length of 1 for a range of small displacements in the y-axis. This is shown by Fig[15] shows that this time taken to diverge has a negative correlation over the initial displacement; it resembles an exponential shape. Fig[16] is the Log(t) where t is the time taken to diverge over the initial displacement; it resembles a straight line. This hints that the initial displacement and time taken to diverge has a negative exponential relationship.

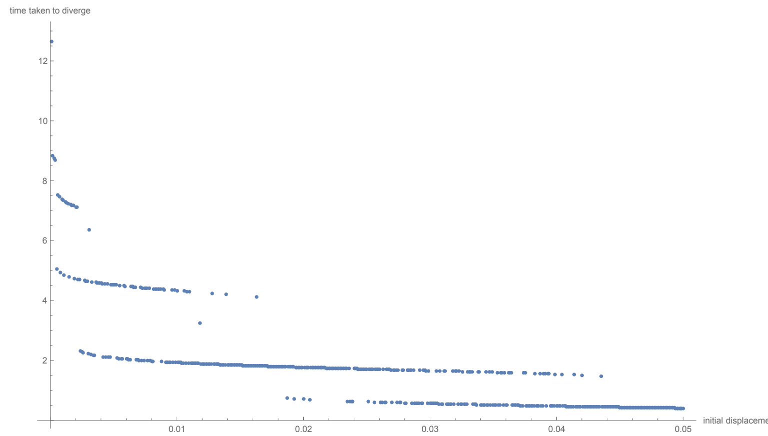

Figure 15: Time taken to diverge over initial displacement graph, using Mathematica.

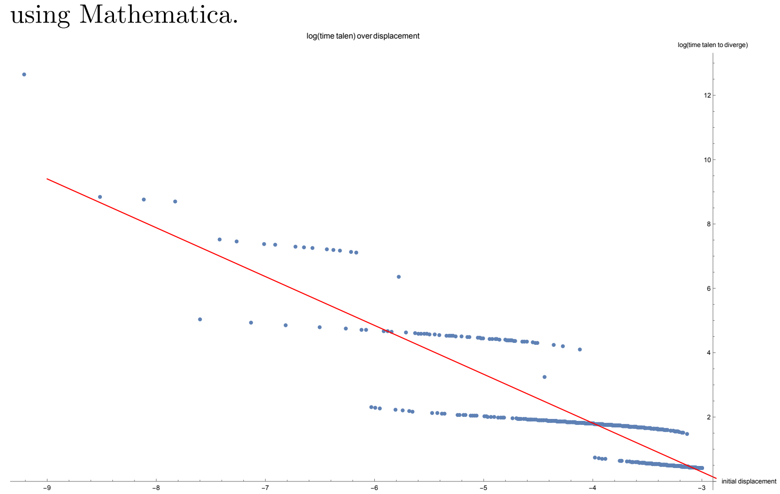

Figure 16: log of time taken to diverge over initial displacement graph using Mathematica.

In order to prove that it is a chaotic system we measured the Lyapunov exponent of the lorenz attractor. The Lyapunov exponent is the measure of rate of separation of trajectories and is defined by equation[6].
[16]

$$\lambda = \lim_{n \to \inf} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_i - t_0 i} ln|\frac{d_i}{d_{0i}}| \tag{8}$$

λ = Lyapunov exponent
t_i = resultant time
t_0 = initial time
d_i = resultant distance between two trajectories
d_0 = initial distance between two trajectories

$$d/d_0 \approx e^{\lambda(t-t_0)} \tag{9}$$

From equation[14] we can see that equation[15] is true. Using equation[15] we plotted $Logfracdd_0$ over $(t - t_0)$ to find the Lyapunov exponent λ of the Lorenz attractor when ρ is greater than 1. We found the gradient, λ, to be 0.913616 for the constants of $\sigma = 28$ $\rho = 10$ and $\beta = 8/3$ at the initial position of (-2.8371821627048677, 0.4841789164422333, 24.15832288560378) and (-2.8371821627048677 + 0.0001, 0.4841789164422333, 24.15832288560378). These initial positions were used to reduce the time for the particles to approach the surface of the attractor and so reduce computational power needed.

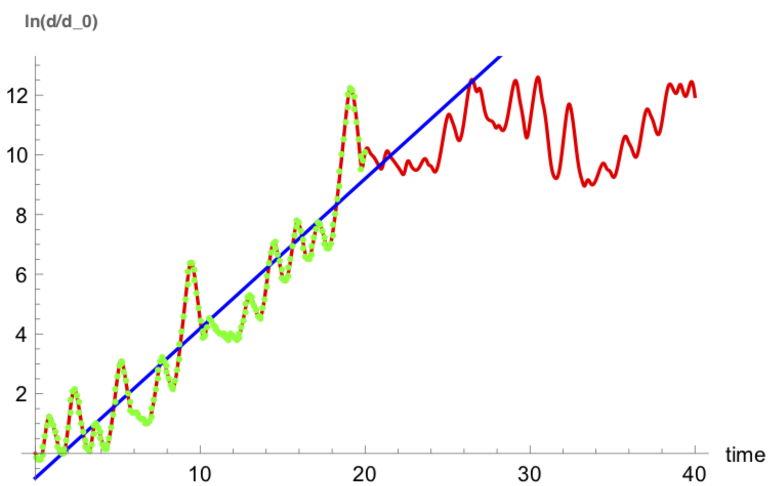

Figure 17: Lyapunov exponent of a chaotic Lorenz system with positive gradient, which means that the two trajectories are diverging, using Mathematica.

We found the λ to be -1.27387 for a system with the constants and initial position set to the same values as in Fig[17] except σ was set to be 0. We also showed that Lorenz system with σ less than 1 had λ less than 0 and therefore it is not a chaotic system as shown by Fig[18].

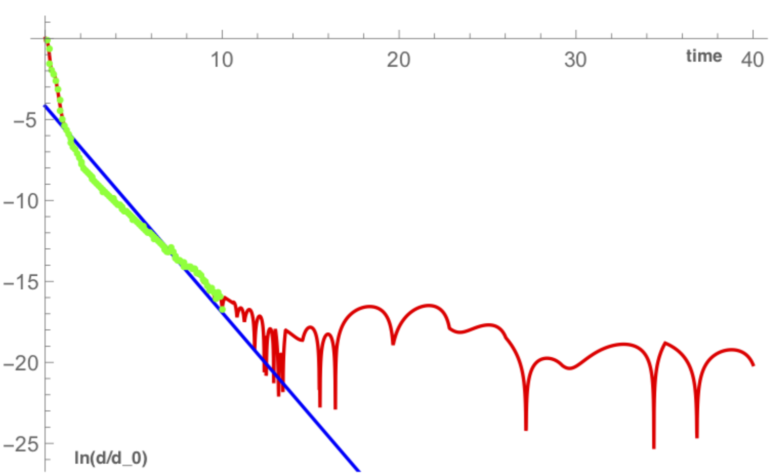

Figure 18: Lyapunov exponent of a simple Lorenz system with negative gradient, which means that the two trajectories are converging, using Mathematica.

Notice in fig[18], the gradient steadily decreases. This is because the of rate separation of trajectories decreases over time and would no longer have a net positive or net negative change when it is at it's "maximum chaotic point". The sensitivity of the Lorenz attractor to the initial conditions and the relationship to fractal dimensions has proved that the Lorenz attractor is indeed chaotic.

5 Practical applications of chaotic systems

The unique properties of a chaotic system is that it can be difficult to predict the resultant state for different initial states due to its sensitiveness. This is a very useful property since it can be used to generate series of numbers that appear as if it is random and is independent of any condition. It is important to emphasise that these series of numbers are not truly random since they will output the exact same values for the same initial conditions but if these initial conditions are even slightly different, they will output a drastically different value.
Used in medicine physics [17]. Found out that Cardiac rhythm is sensitive to initial reactions and is also a fractal. High heart rate means the heart is less likely capable to adapt to commands which leads to heart attack liability. Additionally, for brain treatments. Related by electroencephalographic(EEG) electrical activity in the brain. A healthy brain has high EEG and when resting you will have low EEG. Found that EEG is a strange attractor. Has resulted in developing treatment for a coma.
In weather prediction the fractal structure could be plotted using a computer with many iterations to find periodic orbits with high accuracy essentially helping to predict the unpredictable and complex phenomena seen in the weather and other complex systems.
There are many more such examples of uses of chaos in the world as it allows us to predict complex and seemingly impossible systems.

6 Conclusion

We researched and analysed the behaviour of the Lorenz attractor by using the Lorenz equations on Mathematica[1] and Grapher[2] on MacOS. By first understanding the idea of chaos and an 'attractor' we plotted graphs ranging from simple attractors to the more complex Lorenz Attractor. We concluded that the Lorenz system of equations will always have an attractor. Upon placing particles in the system we saw that it is a global attractor when $\rho < 1$ and the particle will always tend to a point; if $\rho > 1$ it is a strange attractor and the particle will always tend to a surface. In particular we investigated the time taken for two trajectories to diverge significantly for a Lorenz attractor as a function of the closeness of the start points: to find the dependence of the initial starting positions on the time t taken to diverge significantly we wrote a script in Mathematica that computes the time taken for the two norm of the trajectories to reach a unit length of 1 for a range of small displacements in the y-axis. We showed that the initial displacement of the point s and time taken to diverge has a negative exponential relationship. To prove the chaotic behaviour we first researched the relationship to fractals and by finding its fractal dimension using the Hausdorff equation to be 2.06 ± 0.01 we found that the attractor is strange so it may have chaotic behaviour; we found that the Lyapunov exponent could be used to prove if the Lorenz attractor was chaotic by determining how sensitive the Lorenz attractor is to initial conditions. It is very sensitive to the initial conditions as the exponent was $\lambda = 0.913616$ and so it is chaotic as it is a positive value. Non-chaotic behaviour was shown when $\rho < 1$ and we found the Lyapunov exponent to be $\lambda = -1.27387$ supporting the argument that it is not chaotic for this value as the exponent is negative. We have therefore concluded that chaos theory is a vital field of research that allows us to predict complex and seemingly impossible systems.

References

[1] Mathematica, Wolfram Mathematica, https://www.wolfram.com/mathematica/

[2] Grapher, Apple https://support.apple.com/en-gb/guide/grapher/welcome-gcalb3dec608/mac

[3] Deterministic Chaos, The Exploratorium, 1996 https://www.exploratorium.edu/complexity/CompLexicon/chaos.html

[4] Danforth, Christopher M. (April 2013). "Chaos in an Atmosphere Hanging on a Wall". Mathematics of Planet Earth 2013. http://mpe2013.org/2013/03/17/chaos-in-an-atmosphere-hanging-on-a-wall/

[5] http://www.scienceclarified.com/Ca-Ch/Chaos-Theory.html#ixzz59Xw55Iff

[6] Boeing (2015). "Chaos Theory and the Logistic Map". http://geoffboeing.com/2015/03/chaos-theory-logistic-map/

[7] http://www.stsci.edu/~lbradley/seminar/butterfly.html, 2010 Larry Bradley.

[8] Berg, Pierre; Pomeau, Yves; Vidal, Christian (1984). Order within Chaos: Towards a Deterministic Approach to Turbulence. New York: John Wiley & Sons. ISBN 978-0-471-84967-4.

[9] http://www.dictionary.com/browse/attractor, Dictionary.com Unabridged Based on the Random House Dictionary, Random House, Inc. 2018.

[10] https://fractalfoundation.org/resources/what-are-fractals/

[11] http://www.stsci.edu/~lbradley/seminar/fractals.html

[12] http://www.math.lsa.umich.edu/~divakar/Viswanath2004-LorenzFractal.pdf

[13] Afrajmovich, V. S.; Arnold, V. I.; et al. (1994). Bifurcation Theory and Catastrophe Theory. ISBN 3-540-65379-1.

[14] Strange Attractors, 2010 Larry Bradley. http://www.stsci.edu/~lbradley/seminar/attractors.html

[15] Hirsch, Morris W.; Smale, Stephen; Devaney, Robert (2003). Differential Equations, Dynamical Systems, & An Introduction to Chaos (Second ed.). Boston, MA: Academic Press. ISBN 978-0-12-349703-1.

[16] V.L.Kharitonova; A.P.Zhabkob; (2003). LyapunovKrasovskii approach to the robust stability analysis of time-delay systems (Volume 39, Issue 1) ELSEVIER.

[17] A history of chaos theory, PMC https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3202497/